Projection Constraint Lab: Constraint Without Collapse

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Abstract

We study constraint systems that reduce a space without forcing collapse. Two realizations are analyzed: cosine-similarity thresholds in high-dimensional geometry and modular exclusion in arithmetic. Geometric thresholds produce rapid rejection under concentration of measure, while bounded modular filters preserve exact positive density. These behaviors exhibit a shared structure: strong constraint can remove most of a space while preserving structured survivors.

1 Introduction

Constraint systems act by removing admissible configurations according to a rule. In high-dimensional geometry, cosine thresholds define directional filters. In arithmetic, modular exclusions remove residue classes.

These systems appear different but share a common feature:

constraint reduces space without necessarily inducing collapse.

We compare:

- geometric filtering via cosine thresholds,
- arithmetic filtering via modular exclusion.

For background, see Vershynin [2] for high-dimensional concentration and Montgomery and Vaughan [1] for modular density.

2 Geometric Threshold Constraint

Let $x, y \in \mathbb{R}^d$ with cosine similarity

$$\cos\theta = \frac{x \cdot y}{\|x\| \, \|y\|}.$$

We impose the threshold

$$\cos\theta \geq \frac{1}{\sqrt{2}}.$$

In high dimension:

- random vectors concentrate near $\cos\theta \approx 0$,
- fixed thresholds become increasingly selective,
- admissible configurations shrink rapidly.

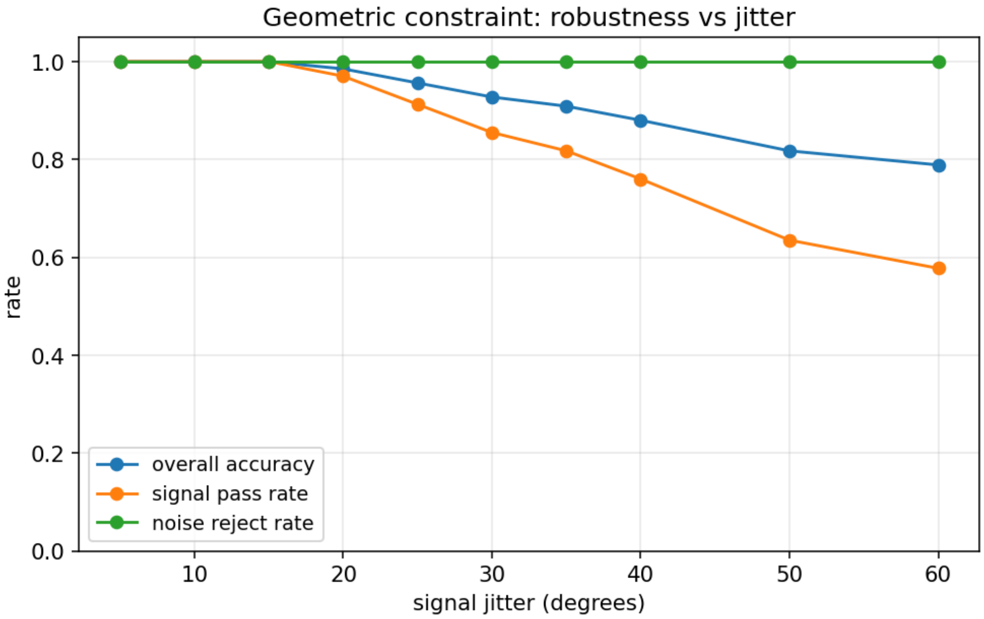


Figure 1: Signal robustness under angular perturbation. The threshold defines a continuous admissibility boundary.

2.1 Scaling Law

Let noise scale per coordinate.

- Fixed noise: mean cosine decays with d.
- Scaled noise ($\sim 1/\sqrt{d}$): mean cosine remains stable.

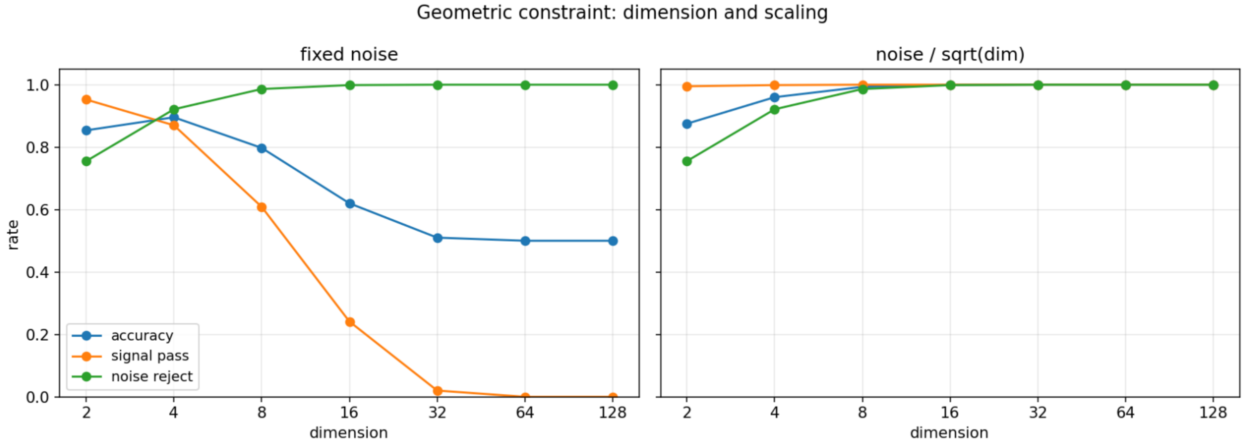


Figure 2: Dimension scaling. Stability is preserved when perturbations scale as $1/\sqrt{d}$.

Thus,

$$\text{signal survival} \iff \text{noise} \sim \frac{1}{\sqrt{d}}. \tag{1}$$

3 Arithmetic Constraint

Consider modular exclusion in $\mathbb{Z}$.

Removing one residue class modulo m yields retained density:

$$1 - \frac{1}{m}.$$

Examples:

$$\frac{4}{5}, \qquad \frac{24}{25}, \qquad \frac{48}{49}.$$

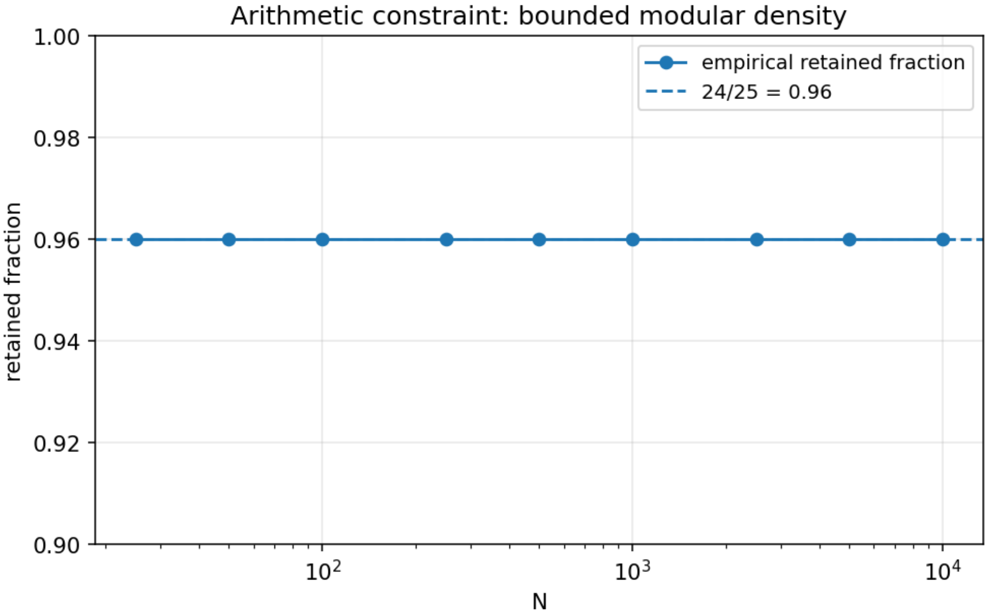


Figure 3: Bounded modular exclusion preserves exact positive density.

This is a discrete analogue of a bounded constraint.

3.1 Cumulative Constraints

Applying multiple exclusions,

$$\prod_{p} \left(1 - \frac{1}{p}\right),$$

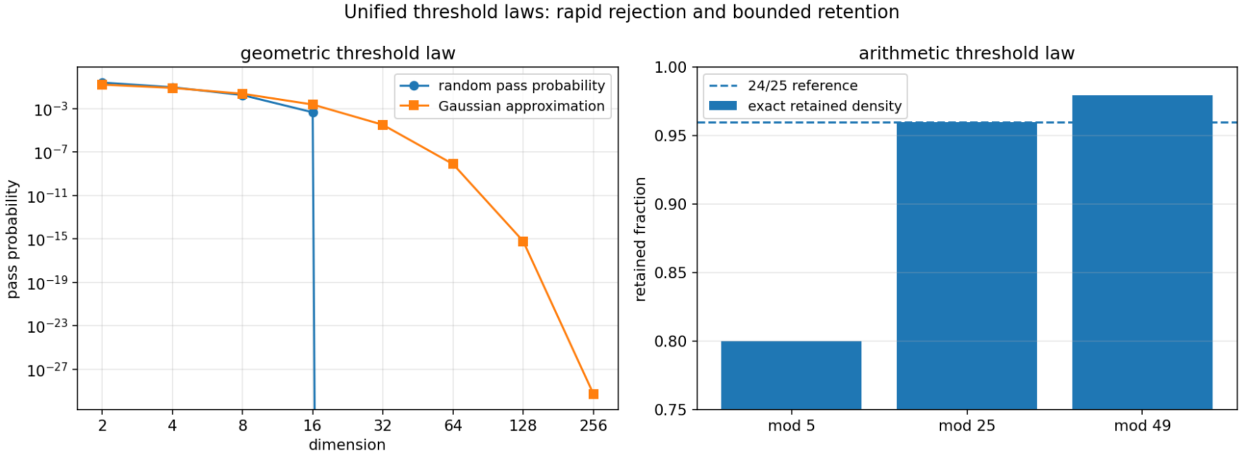
produces multiplicative decay, analogous to geometric rejection.

4 Unified View

We identify a shared structure:

- geometric: threshold filtering in $\mathbb{R}^d$,
- arithmetic: exclusion filtering in $\mathbb{Z}$.

Both systems:


Unified threshold laws: rapid rejection and bounded retention

geometric threshold law

arithmetic threshold law

Figure 4: Geometric rejection vs arithmetic retention.

- reduce admissible space,
- preserve structured survivors under bounded conditions,
- collapse only under unbounded accumulation.

5 Conclusion

Two regimes emerge:

- high-dimensional geometry → rapid rejection,
- bounded modular arithmetic → exact retention.

These illustrate a common principle:

$$\boxed{\text{constraint} \neq \text{collapse}}$$

Computational Source

Figures are generated from reproducible notebooks in the repository:
`https://github.com/thinkthoughts/projection-constraint-lab`

References

[1] Hugh L. Montgomery and Robert C. Vaughan. *Multiplicative Number Theory I: Classical Theory*. Cambridge University Press, 2007.

[2] Roman Vershynin. *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge University Press, 2018.